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A non-classic optimality condition in the problem of control by boundary value conditions of a semi-linear hyperbolic system

A.V. Arguchintsev*

Irkutsk State University, 1, K. Marx Street, Irkutsk 664003, Russia

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A non-classic optimality condition of variational maximum principle type is presented for optimal control problems by initial-boundary conditions of first-order hyperbolic systems. The optimal starting or boundary control provides the maximum in special problems of control by initial values of a system of ordinary differential equations. The optimality condition is illustrated by an example.

Keywords: optimal control; hyperbolic systems; boundary conditions; necessary optimality condition

AMS Subject Classifications: 49K20; 93C20

1. Introduction

Problems of optimal control by initial-boundary conditions of first-order hyperbolic systems arise in mathematical modelling of generation and spreading of waves, processes of chemical technology and population dynamics, etc.

In [3,4] the validity of the differential (linearized) maximum principle as a necessary condition for optimality of boundary controls in first-order hyperbolic systems has been proved. The author proved [1] the maximum principle for the case where the boundary value conditions for hyperbolic equations are being determined by the controlled systems of ordinary differential equations. The characteristic property of a general control by boundary value conditions problem is the fact that in this system there are no optimality conditions similar to the classical (pointwise) maximum principle. Such a peculiarity is emphasized, for instance, in [7], where a counterexample had been constructed for the simplest hyperbolic systems with two orthogonal families of characteristics.

In the present article, the analysis of an increment formula for the cost functional leads to a non-classical optimality condition. The optimal starting or boundary control provides the maximum in special problems of control by initial values of a system of ordinary differential equations. This optimality condition is stronger than the differential maximum principle. The obtained result is similar, in its form,

*Email: arguch@math.isu.ru

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to the variational maximum principle proved in [6] for hyperbolic equations with distributed controls. The optimality condition is proved in terms of the study of perturbation of solution, caused by usual needle variation of control. The concluding part deals with an example which illustrates the optimality condition.

2. Problem statement

We consider an optimal control problem for the following system of semi-linear hyperbolic equations

$$\frac{\partial x}{\partial t} + \mathbf{A}(s,t)\frac{\partial x}{\partial s} = f(x,s,t). \tag{1}$$

The problem is considered in the rectangle $P = S \times T$, $S = [s_0, s_1]$, $T = [t_0, t_1]$. Here $x = x(s,t)$ is an n-dimensional vector-function of state variables, $\mathbf{A} = \mathbf{A}\ (s,t)$ is $n \times n$ — matrix, $(s,t) \in P$.

The system (1) is written in invariant form, i.e. $\mathbf{A}$ is a diagonal matrix. In addition, we assume that the diagonal elements $a_i(s,t)$ of the matrix of coefficients possess constant signs in the rectangle P:

$$a_i(s,t) < 0, \quad i = 1, 2, \ldots, m_1;$$

$$a_i(s,t) = 0, \quad i = m_1 + 1, m_1 + 2, \ldots, m_2 - 1;$$

$$a_i(s,t) > 0, \quad i = m_2, m_2 + 1, \ldots, n.$$

Respectively, the state vector $x = x(s,t)$ contains two subvectors

$$x^- = (x_1, x_2, \ldots, x_{m_1}), \quad x^+ = (x_{m_2}, x_{m_2+1}, \ldots, x_n),$$

which correspond to negative and positive diagonal elements of the matrix of coefficients.

Let the controlled initial-boundary conditions for system (1) be given in the following form:

$$x(s, t_0) = p(u(s), s), \quad s \in S; \tag{2}$$

$$x^+(s_0, t) = g^{(1)}(u^{(1)}(t), t),$$

$$x^-(s_1, t) = g^{(2)}(u^{(2)}(t), t), \quad t \in T. \tag{3}$$

Control functions $u = u(s)$, $u^{(1)} = u^{(1)}(t)$ and $u^{(2)} = u^{(2)}(t)$ are bounded and measurable on segments S and T, respectively, and almost everywhere on these segments the following conditions are satisfied:

$$u(s) \in U \subset E^r; \quad u^{(1)} \in U^{(1)} \subset E^{r_1};$$

$$u^{(2)} \in U^{(2)} \subset E^{r_2}. \tag{4}$$

The problem is to minimize the functional

$$J(u) = \int_S \varphi(x(s,t_1),s)\,\mathrm{d}s + \int_T [\varphi_0(x^-(s_0,t),t) + \varphi_1(x^+(s_1,t),t)]\,\mathrm{d}t + \int_P\int \Phi(x,s,t)\,\mathrm{d}s\,\mathrm{d}t. \tag{5}$$

In order to simplify our notation we introduce the following function:

$$F(x,s,t) = \begin{cases} \varphi_0(x^-(s_0,t),t), & t \in T,\ s = s_0; \\ \varphi(x(s,t_1),s), & s \in S,\ t = t_1; \\ -\varphi_1(x^+(s_1,t),t), & t \in T,\ s = s_1. \end{cases}$$

The optimal control problem (1)–(5) is considered under the following suppositions:

(1) the diagonal elements $a_i = a_i(s,t)$ of the matrix $\mathbf{A}$ are continuous and continuously differentiable in P; in order to avoid awkward notations we suppose that any two functions $a_i = a_i(s,t)$ and $a_j = a_j(s,t)$, $i \neq j$ are being everywhere distinct in P;
(2) the vector-function $p = p(u,s)$ is continuous as a function of u, bounded and measurable as a function of s;
(3) the vector-functions $g^{(1)} = g^{(1)}(u^{(1)}(t),\ t)$ and $g^{(2)} = g^{(2)}(u^{(2)}(t),\ t)$ are continuous with respect to control variables, bounded and measurable with respect to t;
(4) the vector-function $f = f(x,s,t)$ and the scalar functions $\Phi = \Phi(x,s,t)$ and $F = F(x,s,t)$ are continuous with respect to their arguments, and they have continuous and bounded partial derivatives with respect to components of the state vector.

It is suitable to use the definition of a generalized solution in terms of characteristics of the system. Let us consider characteristic curves determined by the ordinary differential equations

$$\frac{\mathrm{d}s}{\mathrm{d}t} = a_i(s,t), \qquad i = 1,2,\ldots,n. \tag{6}$$

Let $s_i = s_i(\xi,\tau;\ t)$ be a solution of Equation (6), which passes through the point $(\xi,\tau) \in P$. If there exists a classical solution of the system under consideration, then the given system is equivalent to the following one:

$$x_i(s,t) = x_i(\xi_i,\tau_i) + \int_{\tau_i}^{t} f_i(x,\xi,\tau)\big|_{\xi = s_i(s,t;\tau)}\,\mathrm{d}\tau, \tag{7}$$

where (ξ_i, τ_i) is the initial point of i-th characteristic curve passing through (s,t).

By means of integral system (7) it is possible to prove the existence and uniqueness of a measurable and almost everywhere bounded in P weak solution [5]. So, instead of the left side of system (1) we consider the differential operator

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_A = \left(\left(\frac{\mathrm{d}x_1}{\mathrm{d}t}\right)_A, \left(\frac{\mathrm{d}x_2}{\mathrm{d}t}\right)_A, \ldots, \left(\frac{\mathrm{d}x_n}{\mathrm{d}t}\right)_A\right),$$

where $(\mathrm{d}x_i/\mathrm{d}t)_A$ is the derivative of i-th component of the state vector along the corresponding family of characteristic curves.

3. Necessary optimality condition

To make the notation more compact, we need an operation of removal of the i-th component of an arbitrary vector of the space E^n. Let $a \in E^n$ and $b_i \in E^1$. We denote

$$\tilde{a}^i = (a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n),$$

$$a \dot{+} b_i = (a_1, a_2, \dots, a_{i-1}, a_i + b_i, a_{i+1}, \dots, a_n).$$

Consider a case of starting control $u = u(s)$.

THEOREM 3.1 *Let the process $\{u, x\}$ be optimal for problem* (1)–(5). *Then almost everywhere on the segment S the following maximum condition is valid*:

$$J(u(\xi), \xi) = \max_{v \in U} J(v, \xi), \qquad \xi \in S, \tag{8}$$

where

$$J(v, \xi) = \sum_{i=1}^{n} \Bigg\{ -F(z^i(\tau_i), \xi_i, \tau_i) \frac{\partial s_i(\xi, \tau_0; \tau_i)}{\partial \xi} \mu(\xi_i, \tau_i) + \int_{t_0}^{\tau_i} [\langle \tilde{\psi}^i(s, t), \tilde{f}^i(z^i(t), s, t) \rangle - \Phi(z^i(t), s, t)]\big|_{s = s_i(\xi,\, t_0;\, \tau)} \frac{\partial s_i(\xi, t_0; t)}{\partial \xi} \, \mathrm{d}t \Bigg\}.$$

Here $\langle ., . \rangle$ is a designation of a scalar product in E^{n-1},

$$z^i(t) = x(s_i(\xi, t_0; t), t) \dot{+} (y_i(t) - x_i(s_i(\xi, t_0; t), t)),$$

the functions $y_i(t)$ are defined by ordinary differential equations

$$\dot{y}_i(t) = f_i(z_i(t), s_i(\xi, t_0; t), t), \tag{9}$$

$$t \in [t_0; \tau_i]; \qquad y_i(t_0) = p_i(v, \xi), \tag{10}$$

(ξ_i, τ_i) are the end points for characteristic curves $s = s_i(\xi, t_0; t)$, and $\psi = \psi(s, t)$ is a solution of the conjugate problem

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)_A + \mathbf{A}_s(s, t)\psi = -H_x(\psi, x, s, t), \quad (s, t) \in P;$$

$$\psi(s, t_1) = -\varphi_x(x(s, t_1), s), \quad s \in S; \tag{11}$$

$$\psi_i(s_0, t) = \frac{1}{\lambda_i(s_0, t)} \frac{\partial \varphi_0(x^-(s_0, t), t)}{\partial x_i},$$

$$i = 1, 2, \dots, m_1;$$

$$\psi_i(s_1, t) = -\frac{1}{\lambda_i(s_1, t)} \frac{\partial \varphi_1(x^+(s_1, t), t)}{\partial x_i},$$

$$i = m_2, m_2 + 1, \ldots, n;$$

on the considered optimal process,

$$\mu(\xi_i, \tau_i) = \begin{cases} 1, & \tau_i = t_1, \quad s_0 < \xi_i \leq s_1; \\ -1/\lambda_i(s_0, \tau_i), & t_0 < \tau_i \leq t_1, \ \xi_i = s_0; \\ -1/\lambda_i(s_1, \tau_i), & t_0 < \tau_i \leq t_1, \ \xi_i = s_1. \end{cases}$$

In a similar way this theorem can be formulated for boundary controls $u^{(1)} = u^{(1)}(t)$, $u^{(2)} = u^{(2)}(t)$.

Proof of Theorem 3.1 is performed by means of study of the cost functional increment formula on needle variation of the form

$$\Delta u(s) = \begin{cases} v - u(s), & s \in S_\varepsilon; \\ 0, & s \in S \backslash S_\varepsilon. \end{cases}$$

Here $S_\varepsilon = (\xi - \varepsilon, \xi]$, the point $\xi \in (s_0, s_1]$, the value $\varepsilon \in (0, \xi - s_0]$ and $v \in \mathrm{U}$.

The estimation

$$\|\Delta x(s, t)\|_{E^n} \leq K\varepsilon, \quad K > 0,$$

of the state increment for the considered variation is valid only for points which do not belong to the characteristic bands

$$\{(s, t) \in P : s_i(\xi - \varepsilon, t_0; t) < s \leq s_i(\xi, t_0; t)\}.$$

It is impossible to estimate in a similar manner components $\Delta x_i(s, t)$ of state increments in the corresponding characteristic bands in terms of parameter of measure of needle variation domain. Impossibility to prove analogues of classical Pontryagin's maximum principle is to be explained by just this circumstance.

The difference of further stages of the proof from the standard technique applied to obtain necessary optimality conditions of first-order consists [2] in the following:

- in the increment formula the terms connected with increments of components of the state vector in corresponding characteristic bands are separated;
- by means of the change of integration variables we pass from integration over segments containing endpoints of characteristics, which supports on intervals of needle variation, to the integration over these segments of variation;
- functions $y_i(t)$, which can be calculated in terms of data of the problem for characteristic curves only, is introduced instead of components Δx_i in i-th characteristic bands; so, we eliminate implicit dependence of components Δx_i on characteristic curves $s_i(\eta, t_0; t)$ and on the rest of the components of this increment. $\square$

The proved theorem is a stronger necessary optimality condition in comparison with the differential maximum principle.

4. An illustrative example

In the rectangle $S \times T$, $S = [s_n, s_k]$, $T = [t_0, t_1]$ consider the following initial-boundary value problem

$$x_{1t} + \lambda x_{1s} = \alpha(s)(x_1 - x_2),$$

$$x_{2t} - \lambda x_{2s} = \beta(s)(x_1 - x_2),$$

$$x_1(s, t_0) = u(s) + q(s), \qquad x_2(s, t_0) = u(s) - q(s), \qquad s \in S.$$

Admissible controls are supposed to be scalar functions $u(s)$, which satisfy the condition

$$u(s) \in U \subset E^1, \qquad s \in S.$$

The problem is to minimize the functional

$$J(u) = \int_S (x_1(s, t_1) + x_2(s, t_1) - \eta(s))^2 \, \mathrm{d}s.$$

The functions $\alpha(s)$, $\beta(s)$, $q(s)$, $\eta(s)$ and the positive constant λ are supposed to be given. In addition, we propose that the condition

$$s_k - s_n > 2\lambda(t_1 - t_0), \tag{12}$$

is valid. Its meaning will be explained later.

Two characteristic families are determined by the equations $s_1 = -\lambda t + \text{const}$, $s_2 = \lambda t + \text{const}$. Here

$$\frac{\partial s_1(\xi, t_0; \alpha)}{\partial \xi} = \frac{\partial s_2(\xi, t_0; \alpha)}{\partial \xi} = 1.$$

Let $\{u, x\}$ be an optimal process and $\psi(s, t)$ be a corresponding solution of the conjugate problem

$$\psi_{1t} + \lambda \psi_{1s} = -\alpha\psi_1 - \beta\psi_2,$$

$$\psi_{2t} - \lambda \psi_{2s} = \alpha\psi_1 + \beta\psi_2,$$

$$\psi_1(s, t_1) = \psi_2(s, t_1) = 2(\eta(s) - x_1(s, t_1) - x_2(s, t_1)), \qquad s \in S,$$

$$\psi_1(s_k, t) = \psi_2(s_n, t) = 0, \qquad t \in T.$$

It follows from Theorem 3.1 that the maximum condition (11) of the functional $I(v, \xi)$ holds almost everywhere in S. A form of this functional and a corresponding system of ordinary differential equations depend on the arrangement of a point ξ.

(a) $s_n + (t_1 - t_0)\lambda \le \xi \le s_k - (t_1 - t_0)\lambda$.

The inequality (12) guarantees that this segment is non-empty. Equations of characteristic curves originated from the point (ξ, t_0) are the following:

$$s - 1 = \lambda(t_0 - t) + \xi, \qquad s_2 = \lambda(t - t_0) + \xi.$$

The finishing points of these characteristics are the points of a segment $\{(s, t)$: $s \in S, t = t_1\}$. In our case

$$\begin{aligned}
I(v, \xi) = &-(y_1(t_1) + x_2(\lambda(t_0 - t_1) + \xi, t_1) - \eta(\lambda(t_0 - t_1) + \xi))^2 \\
&+ \int_{t_0}^{t_1} \psi_2(\lambda(t_0 - t) + \xi, t)\, \beta(\lambda(t_0 - t) + \xi)(y_1(t) - x_2(\lambda(t_0 - t) + \xi, t))\, \mathrm{d}t \\
&- (y_2(t_1) + x_1(\lambda(t_1 - t_0) + \xi, t_1) - \eta(\lambda(t_1 - t_0) + \xi))^2 \\
&+ \int_{t_0}^{t_1} \psi_1(\lambda(t - t_0) + \xi, t)\, \alpha(\lambda(t - t_0) + \xi)(x_1(\lambda(t - t_0) + \xi, t) - y_2(t))\, \mathrm{d}t,
\end{aligned}$$

$$\dot{y}_1(t) = \alpha(\lambda(t_0 - t) + \xi)(y_1(t) - x_2(\lambda(t_0 - t) + \xi, t)), \qquad t \in T,$$

$$\dot{y}_2(t) = \beta(\lambda(t - t_0) + \xi)(x_1(\lambda(t_0 - t) + \xi, t) - y_2(t)), \qquad t \in T,$$

$$y_1(t_0) = v(\xi) + q(\xi), \qquad y_2(t_0) = v(\xi) - q(\xi).$$

(b) $s_n \le \xi \le s_n + (t_1 - t_0)\lambda$.

In this case a characteristic of the first family passes through a point (ξ, t_0) and has $(s_n, t_0 + (\xi - s_n)/\lambda)$ as a finishing point. A characteristic of the second family finishes in a point $(\lambda(t_1 - t_0) + \xi, t_1)$. The cost functional is

$$\begin{aligned}
I(v, \xi) = &\int_{t_0}^{t_0 + (\xi - s_n)/\lambda} \psi_2(\lambda(t_0 - t) + \xi, t)\, \beta(\lambda(t_0 - t) + \xi)(y_1(t) \\
&- x_2(\lambda(t_0 - t) + \xi, t))\, \mathrm{d}t - (y_2(t_1) + x_1(\lambda(t_1 - t_0) + \xi, t_1) - \eta(\lambda(t_1 - t_0) + \xi))^2 \\
&+ \int_{t_0}^{t_1} \psi_1(\lambda(t - t_0) + \xi, t)\, \alpha(\lambda(t - t_0) + \xi)(x_1(\lambda(t - t_0) + \xi, t) - y_2(t))\, \mathrm{d}t.
\end{aligned}$$

Here functions $y_1(t)$ and $y_2(t)$ are solutions of Cauchy problem (9) and (10). However, Equation (9) is considered for $t \in [t_0, t_0 + (\xi - s_n)/\lambda]$.

(c) $s_k - (t_1 - t_0)\lambda < \xi \le s_k$.

This variant is a symmetric to the just-considered case. A characteristics of the second family originated from the point (ξ, t_0) has a point $(s_k, t_0 + (s_k - \xi)/\lambda)$ as a finishing point. A finishing point of a characteristics of the first family is $(\lambda(t_0 - t_1) + \xi, t_1)$.

$$\begin{aligned}
I(v, \xi) = &-(y_1(t_1) + x_2(\lambda(t_0 - t_1) + \xi, t_1) - \eta(\lambda(t_0 - t_1) + \xi))^2 \\
&+ \int_{t_0}^{t_1} \psi_2(\lambda(t_0 - t) + \xi, t)\, \beta(\lambda(t_0 - t) + \xi)(y_1(t) - x_2(\lambda(t_0 - t) + \xi, t))\mathrm{d}t \\
&+ \int_{t_0}^{t_0 + (s_k - \xi)/\lambda} \psi_1(\lambda(t - t_0) + \xi, t)\, \alpha(\lambda(t - t_0) + \xi)(x_1(\lambda(t - t_0) + \xi, t) - y_2(t))\, \mathrm{d}t.
\end{aligned}$$

Functions $y_1(t)$ and $y_2(t)$ are solutions of Cauchy problem (9) and (10). However, Equation (9) is considered for $t \in [t_0, t_0 + (s_k - \xi)/\lambda]$.

This example illustrates a variant of different diagonal elements of a matrix of the hyperbolic operator. The corresponding optimal control problems by ordinary differential equations has a number of distinctive features.

First of all, in spite of additivity of the cost functional each problem does not separate into optimal control problems constructed along the corresponding characteristic families. Secondly, each equation of an ordinary differential system is considered, generally speaking, for different segments of independent variable.

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