Forward and Futures Contracts

Part I. Forward Contracts

1. Contract design and trading mechanics.

2. Finding forward price by an arbitrage argument: creating a synthetic forward.

3. Finding PV of a seasoned forward position (marking to market a previously initiated position).

   We will consider three cases:

   1. Underlying asset provides no income prior to maturity of the forward contract

   2. Known cash dividends

   3. Known percentage dividend yield

4. Currency forwards

5. Forward Rate Agreements (FRAs) – interest rate forwards

Contracts and Trading Mechanics

A **forward contract** is a contract to buy or sell an underlying asset at a predetermined price **K** (**delivery price**) on a specified future date T.

- **Long party** agrees to buy the underlying asset at the delivery price K at time T.
- **Short party** agrees to sell the underlying asset at the delivery price K at time T.

- **Settlement:** The contract is settled at maturity T: the short delivers the asset to the long in return for cash amount K.
- **Counterparties:** bilateral over-the-counter (OTC) contracts negotiated between two counterparties (between two financial institutions or between a financial institution and its customer).
- **Payoff at Expiration:**
  - **The long** receives the asset worth S_T (price of the asset at maturity of the forward T) and pays the delivery price K. Thus the cash flow (payoff) from the long forward position at maturity T is: $S_T - K$. Some forward contracts are cash physically settled, while some are cash

settled.

- **The short** receives the cash amount K and delivers the asset worth S_T in exchange (payoff is $K - S_T$).

Valuation Problem: how do you establish the delivery price K in the forward contract?

Notations:

- S_t : underlying asset price at time t ;
- K : delivery price specified at contract inception;
- $V(t,T)$: present value at time $t \geq 0$ of the forward contract initiated at $t=0$ and expiring at T ;
- $F(t,T)$ or simply F : forward price at time t for settlement at time T ;
- r - risk-free interest rate per annum with continuous compounding (initially we assume that the term structure is flat);

We will use the following modeling **assumptions of perfect markets**:

- Market participants can lend and borrow at the same risk-free rate of interest.
- There are no arbitrage opportunities, or market participants take advantage of arbitrage opportunities instantaneously as they occur.

Case 1: Forward Contract on an Investment Asset that Pays No Dividends

- Payoff from a long forward position: $V(T,T) = S_T - K$
- At inception, a forward contract is neither an asset nor liability. It is set up so that its PV at inception is zero to both parties: $V(0,T) = 0$

- **Definition:** Forward price $F(0,T)$ is such a delivery price K that sets the time-zero present value of the forward contract to zero.
- **Problem:** find such $K = F(0,T)$ that sets the PV of the forward contract to zero, i.e., $V(0,T) = 0$.

- **Result:** $K = F(0,T) = e^{rT} S$
- **Proof:** creating a **synthetic forward position** and **arbitraging** the actual forward contract against the synthetic forward position.
  - **Synthetic forward position** ($t=0$)**:**
    - Borrow $e^{-rT} K$ dollars until time T at the risk-free rate r ;
    - Buy the asset at spot price S;
  - Payoff at time T is $S_T - K$, which is the same as the payoff of the forward contract. Therefore, its value (price) at $t=0$ should be equal to the value of forward contract (to avoid arbitrage). The value of synthetic forward at time zero is $S_0 - e^{-rT} K$.
  - At strike level $K^* = e^{rT} S_0$, the value of forward contract is zero. Therefore, forward price $F(0,T) = e^{rT} S$.

- Suppose $F(0,T) = e^{rT} S$ does not hold, and for example, $F(0,T) > e^{rT} S$. This violates non-arbitrage assumption. To realize arbitrage:

  Short the actual forward and go long a synthetic forward (create a synthetic forward contract by borrowing cash and buying the asset):

- At time $t=0$:
    - Borrow S dollars until time T at the risk-free rate r ;
    - Use these dollars to buy the asset at spot price S ;
    - Take a short position in the forward contract with delivery price $F(0,T)$

    Note, no capital investment is needed to do this (zero initial cost).
- At time T :
    - Sell the asset for the delivery price $F(0,T)$;
    - Use the amount $e^{rT}S$ of the proceeds to repay the loan plus interest.
- **Profit and Loss (P&L):** a riskless and costless arbitrage profit of $F(0,T)-e^{rT}S>0$.

- In the case $F(0,T)<e^{rT}S$ the arbitrage can be realize by going **long in the actual forward and shorting a synthetic forward.**
- Thus $F(0,T)=e^{rT}S$

Important note: The second arbitrage is only valid for investment assets.

- Forwards on assets that are held for consumption purposes cannot be valued by this argument. The reason is that it calls for selling the physical asset and replacing it with the long forward position. But there is a **convenience yield** to holding a consumption asset in stock – market participants who need the commodity for consumption will not sell it even if $F(0,T) < e^{rT} S$.
- Thus, for consumption assets we cannot ascertain the equality $F(0,T) = e^{rT} S$. We can only ascertain the inequality $F(0,T) \le e^{rT} S$.

Calculating PV of a seasoned forward position (marking to market a forward)

- At some time t during the life of the contract, $0 \le t \le T$, PV of a long forward contract with delivery price K is generally different from zero. It is equal to the present value of the difference between the current forward price $F(t,T)$ at time t and the delivery price K (that was set equal to the forward price $F(0,T)$ at the forward contract inception $t=0$):

$$V(t,T) = e^{-r(T-t)}(F(t,T) - K)$$

- This relationship is valid for all types of underlying assets.

- **Proof:**
  - Payoff of seasoned contract is $S_T - K$

    Payoff of a newly written forward contract is $S_T - F(t,T)$
  - Since $S_T - K = (S_t - F(t,T)) + (F(t,T) - K)$, the seasoned forward contract is equivalent to the combination of:

    A newly written forward contract + fixed cash amount $(F(t,T) - K)$ to be received at T.
  - Since PV of the newly written forward is zero, PV of the seasoned forward is equal to PV of the cash amount $(F(t,T) - K)$, i.e.,

    $$V(t,T) = e^{-r(T-t)}(F(t,T) - K)$$
  - For the underlying asset that pays no income before time T:

    $$V(t,T) = S_t + e^{-r(T-t)}K$$

Case 2: Forward Contract on an Investment Asset Providing Known Cash Income

Examples: Coupon bearing bonds, stocks paying cash dividends.

- Let I_0 be PV at $t=0$ of all income to be received from the asset between $t=0$ and T (discounted at the risk-rate; assume all future cash dividends between 0 and T are known).
- **Problem:** find forward price $F(0,T)$ such that PV of the forward contract written on the asset is zero at t = 0, i.e. $V(0,T)=0$.
- **Result:** $K = F(0,T) = e^{rT}(S-I_0)$
- **Proof:** Create a synthetic forward position ($t=0$) which replicates the forward contract:
  - Borrow $e^{-rT}K$ dollars until time T at the risk-free rate r;
  - Buy the asset at spot price S;
  - Invest I_0 dollars at the risk-free rate r;
  - Payoff of a synthetic forward position $S_T - K$ (the same as forward payoff)
  - The value of synthetic forward at time zero is $S_0 - e^{-rT}K - I_0$.

- At strike level $K^* = e^{rT}(S_0 - I_0)$, the value of forward contract is zero.

Case 3: Forward Contract on an Investment Asset Providing a Known Dividend Yield

- Suppose the underlying asset pays dividends continuously at the constant rate q (dividend yield), so that the dividend paid over an infinitesimal time period dt is $qS_t dt$.
- Then e^{-qT} units of the asset purchased at time $t=0$ grow to one unit of the asset by time $t=T$ due to the accumulation of dividends, assuming all dividends are reinvested in the asset as soon as they are received.

- **Problem:** find the forward price $F(0,T)$ such that PV of the forward contract is zero at t = 0, i.e., $V(0,T)=0$.
- **Result:** $K = F(0,T) = e^{(r-q)T} S$

- **Proof:** Create a synthetic forward position ($t=0$) which replicates the forward contract:
    - Borrow $e^{-rT}K$ dollars until time T at the risk-free rate r;
    - Buy e^{-qT} units of asset at spot price S ;
  - Payoff of a synthetic forward position $S_T - K$ (the same as forward payoff)
  - The value of synthetic forward at time zero is $e^{-qT}S - e^{-rT}K$
  - At strike level $K^* = e^{(r-q)T}S$, the value of forward contract is zero.

Application: Currency Forwards

- If the underlying asset is a foreign currency, then the forward exchange rate is:

  $$F(0,T) = e^{(r_d - r_f)T} S$$

  where r_d is the domestic risk-free interest rate (with continuous compounding) and r_f is the foreign risk-free interest rate (with continuous compounding). The foreign currency can be thought of as the asset that pays continuous dividend yield at the rate r_f .

- The exchange forward rate can be expressed using the domestic and foreign discount factors $F(0,T) = e^{(r_d - r_f)T} S = \frac{P_f(0,T)}{P_d(0,T)} S$.

Conversion USD funding to RUB funding using currency forward

- Suppose, you are the western bank having access to USD LIBOR market and you need to provide your Russian client with RUB funding (A client needs to borrow N amount of RUB).
- Assume also that USD/RUB forward rate is F and USD LIBOR rate for time T is r_{USD} . Because $F=S(0)\exp[(r_{RUB}-r_{USD})T]$, the implied RUB rate is $r_{RUB}=r_{USD}+\frac{\log(\frac{F}{S(0)})}{T}$. $S(0)$ is the current USD/RUB spot rate.

How to convert USD borrowing to RUB borrowing using Currency Forward?

- **1 step.** *At time* $t=0$ *:* Borrow $\frac{N}{S(0)}$ of dollars on LIBOR at r_{USD} rate. You are supposed to return $\frac{N}{S(0)}\exp(r_{USD}T)$ at time $t=T$.
- **2 step.** *At time* $t=0$ *:* Covert USD to RUB using spot exchange rate $S(0)$ and lend this money (N RUB) to the borrower. A borrower is supposed to return

you $N\exp(r_{RUB}T)$ RUB at time $t=T$.

- **3 step.** *At time* $t=0$ *:* Enter a forward contract to buy $\frac{N}{S(0)}\exp(r_{USD}T)$ USD at time $t=T$.
- **4 step.** *At time* $t=T$ *.* Forward Settlement: You deliver $N\exp(r_{RUB}T)$ RUB and receive $\frac{N}{S(0)}\exp(r_{USD}T)$ USD, and return this USD amount to your lender.

The result: *Being able to borrow on LIBOR and having access to USD/RUB forwards, we are able to lend RUB at* ***implied forward RUB rates****.*

Interest rate forwards

Forward rates $f(t,T,S)$ are interest rates implied by the current spot rates $r(t,T)$ for periods of time in the future.

- Forward rates are characterized by three time instances, namely the current time t, beginning of the forward accrual period T, and the end of forward accrual period S with $t \le T \le S$
- Forward rates are interest rates that can be locked in at current time t for an investment over the forward accrual period (T,S)
- It follows from the absent of arbitrage assumption that the forward rate can be computed as follows:
  - First, calculate the price of forward bond spanning (T,S) :

    $B(T,S)=\frac{P(t,S)}{P(t,T)}$, then from $\frac{1}{1+\tau(T,S)f(t,T,S)}=\frac{P(t,S)}{P(t,T)}$

    compute the value of
  - **forward rate** (simply compounded)

$$f(t,T,S)=\frac{P(t,T)/P(t,S)-1}{\tau(T,S)}$$

- Similar, continuously compounded forward rate

$$f(t,T,S)=\frac{\ln(P(t,T)/P(t,S))}{\tau(T,S)}$$

- Relationship between spot and forward rates $R(t,S)$, $R(t,T)$, $f(t,T,S)$:

$e^{R(t,S)(S-t)}=e^{R(t,T)(T-t)}e^{f(t,T,S)(S-T)}$, or

$R(t,S)(S-t)=R(t,T)(T-t)+f(t,T,S)(S-T)$

Example:

Year	Spot rate, %	Forward for nth year
1	1.0	
2	2.0	3.0
3	3.0	5.0

- **Example:** An investor can lock in the forward rate for a future time period. Suppose, zero rate for 1 and 2 years are $R_1=1\%$, $R_2=2\%$ (continuously compounded). If an investor borrows $100 at 1% and invest money at 2% for 2 years, the result is cash outflow of

$\exp(0.01*1)=\$101.0$ at the end of the 1 year, and inflow of $\exp(0.02*2)=\$104.1$ at the end of the 2 year. Therefore, the return rate is **3%** $\exp(0.03)=\frac{104.1}{101.0}$, which is equal to the forward rate.

Forward Rate Agreement (FRA)

- **FRA** is a OTC contract where party A agrees to lend certain principle to party B at a predetermined interest rate for the period of time starting and ending in the future
- FRA is fully specified by the following contractual parameters:
  1) N is the principle underlying the contract
  2) $T_1<T_2$ - begin and end dates of the contract period
  3) R_K - the rate of interest agreed in the FRA
- Normally. interest rate R_K is expressed using simple compounding convention:

1) At time T_1 party A pays principle amount N to party B

2) At time T_2 party B pays back principle N plus interest R_K

- Interest rate R_K is set in such a way that the present value of the contract is zero no money is exchanges hands at the contract origination time

- **Valuation of FRA:** the value of FRA to party A is equal to

$$V_A(T_t) = -N\,P(t,T_1) + N(1+R_K\,\tau(T_{1,}T_2))P(t,T_2)$$
$$= N\,\tau(T_{1,}T_2)P(t,T_2)\left[R_K - R(T_{1,}T_2)\right]$$

- Thus, the contract rate should be equal to the forward rate

$$R_K = F(t,T_{1,}T_2)$$

Part 2. Futures

Futures contracts vs. forward contracts

Futures contracts are specified by the exchange, while forward contracts are private OTC arrangements.

Forwards	**Futures**
Private OTC contracts	Traded on exchange
Not standardized	Standardized
Specific delivery date and the asset are specified unambiguously	Delivery options: timing option, quality option
Settled at contract expiration	Settled daily
Counterparty credit risk	No credit risk

Designing a new futures contract (more complicated instruments than forwards):

- Specify the underlying asset - quality option and location option

- Contract size
- Delivery arrangements - timing options (for commodities delivery can be made anytime during the delivery month; the short has the right to choose when to deliver)
- Price quotes - minimum price movement
- Daily price movement limits (limit up/limit down)
- Position limits for speculators

Futures Trading Mechanics

- Deposit **initial margin** with your broker (margins set by the exchanges depending on the contract's volatility and changed often).
- **Daily settlement:** at the end of each day your account is marked to market and actually adjusted to reflect the current end-of-day P&L: deposits are made and losses are withdrawn. Futures contracts are actually re-settled each day: PV of a futures position is always zero at the end of the day: V is resettled each day to zero.
- Many financial futures are settled in cash (e.g., S&P 500 futures).

Convergence of futures price to spot

As the delivery month of a futures contract is approached, the futures price converges to spot.

If this were not true, there would be an arbitrage opportunity.

Relationship Between Forward and Futures Prices when Interest Rates are Deterministic

Problem: futures contracts are settled every day, while forward contracts are settled at maturity. Does this result in any differences between futures and forward prices?

Theorem: Futures and forward prices are equal when interest rates are non-stochastic (known in advance).

Appendix in Chapter 5 in Hull gives the proof under the assumption of flat term structure of interest rates. The proof can be extended to deterministic term structure of interest rates that is not necessarily flat.

What happens when interest rates are stochastic?

If asset price changes are **positively correlated with interest rate** changes, then t**he futures price tends to be greater than the forward price**. Gains in futures resulting from the increase in the asset price will be realized when interest rates tend to be higher, thus earning greater interest. Vice versa, if asset price changes are negatively correlated with interest rate changes, the futures price tends to be lower than the forward price. The difference is not significant for short-dated contracts, but becomes significant as maturity increases and the absolute value of the correlation between the asset and interest rates increases.

Commodity Futures

- Investment vs. consumption commodities
- Storage costs
- Cost of carry
- Convenience yield
- Term structure of convenience yields and futures prices

Investment vs. consumption commodities

- **Investment commodities** are commodities held by a significant number of investors solely for investment purposes (gold and silver).
- **Storage costs** can be regarded as negative income. Let U denote the present value of all storage costs to be incurred over the life of the contract: U=PV(All storage cost)
  - Then the **futures price** of an investment commodity (gold, silver) is:

    $$F = e^{rT}(S+U)$$

  - If storage costs are proportional to the price of the commodity, they can be regarded as a negative dividend yield: $F = e^{(r+u)T} S$, where u are storage costs per annum as a proportion of the spot price.

- **Consumption commodities .** There are certain benefits associated with holding physical commodities in stock (keep production running, avoid stock-outs, take advantage of local short-term shortages, etc.)
  - Thus for consumption commodities we only have the inequality:

  $F \le e^{rT}(S+U)$

  - Even if futures fall to a discount, the holders of physical commodity will not sell the physical and buy the futures, since they need the physical commodity for current consumption.
  - Suppose y is an effective payment rate (yield) for the potential inconvenience of a stock- out for commodity, called **convenience yield**. It is defined as follows: $F = e^{(r-y)T}(S+U)$
  - For investment assets, convenience yield should be equal to zero.

Otherwise, there are arbitrage opportunities.

- **Cost of carry** = interest paid to finance the asset + storage costs - income earned on the asset
- Cost of carry rate for different assets:
  - Assets with no dividends and storage costs: r
  - Stock index: $r-q$
  - Foreign currency: $r_d - r_f$
  - Commodity: $r+u$
- Then, the futures price can be expressed as (c – cost of carry rate):
  - Investment asset: $F = e^{cT} S$
  - Consumption asset: $F = e^{(c-y)T} S$